Solutions

Minnesota State High School Mathematics League

Individual Event

2005-06 Event 5A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 20 minutes for this event.

all three must be correct for 1 point

Faces Painted	Number of cubes
1	24
2	24
3	8

1. A cube 4 inches on its edges is painted red. Then, by making cuts parallel to the faces, it is cut into 64 cubes with 1 inch edges. Fill in the table that counts the number of cubes having 1, 2, or 3 faces painted.

10

2. In the two additions,
$$A + B = C$$
$$C + D = EA$$
each of the five letters represents a distinct digit, EA being a two digit number. What is the value of $B+D$?

29

3. With one straight line, you can slice a pie into two pieces; a second cut that crosses the first one will produce four pieces; and a third cut can produce as many as seven pieces (Figure 3). What is the largest number of pieces that you can get with seven straight cuts?

one point for each correct answer

a) Beth

b) Amy

4. (a) Amy, Beth, and Christine toss a coin 15, 16 and 17 times respectively. Which girl is least likely to get more heads than tails?
(b) Amy, Beth, and Christine toss a coin 18, 19 and 20 times respectively. Which girl is least likely to get more heads than tails?

2. Surely $E = 1$

Then $(C) + D = 10 + A$

so $(A+B) + D = 10 + A$

$B + D = 10$

3. The second cut adds two pieces; the third cut adds three. Convince yourself that this is the pattern

Number of cuts	Number of pieces
0	1
1	2
2	4
3	7
4	11
5	16
6	22
7	29

4. (a) For an even number of tosses, the probability of getting the same number of heads and tails reduces the probability of one being tossed more than another. Example with $n = 6$

prob (3 heads) $= \frac{20}{64}$

prob (less than 3) = prob (more than 3) $= \frac{22}{64}$

Beth, with an even number of tosses, is least likely to get more heads than tails.

(b) Note $\begin{cases} \text{prob (4 heads in 8 tosses)} = \frac{70}{256} \\ \text{prob (5 heads in 10 tosses)} = \frac{63}{256} \end{cases}$

The higher the number of even tosses, the lower the probability of same number of heads and tails, so the higher the probability

Solutions

Minnesota State High School Mathematics League
Individual Event

2005-06 Event 5B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

57, 158

$33\sqrt{3}$ 1. An isosceles trapezoid has base angles of 60° and bases of 14 and 8. What is its area?

$\frac{50}{7}$ 2. ΔABC has sides of lengths $a=4$, $b=10$, and $c=7$. $\Delta UVW \sim \Delta ABC$, and ΔUVW has a perimeter of 15. What is the length of the longest side of ΔUVW?
7.142

3. $ABCDEF$ is a regular hexagon with center P and sides of length s. ΔPQR is an equilateral triangle with Q and R exterior to the hexagon as shown in Figure 3. PR intersects AB at U, making $AU = s/3$. Find in terms of s the area of the quadrilateral common to both the triangle and the hexagon.
$\frac{s^2\sqrt{3}}{4}$

$20\sqrt{11}$ 4. In ΔABC (Figure 4), $AB=9$, $AC=15$, and $m(\angle B) = 2m(\angle C)$. What is the area of ΔABC?
66.3325
66.332

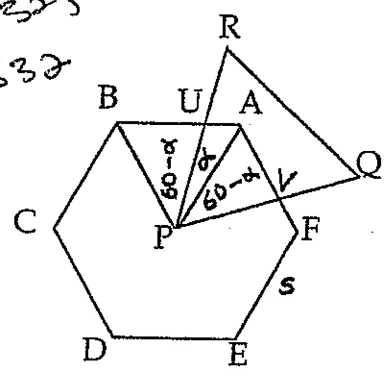


Figure 3

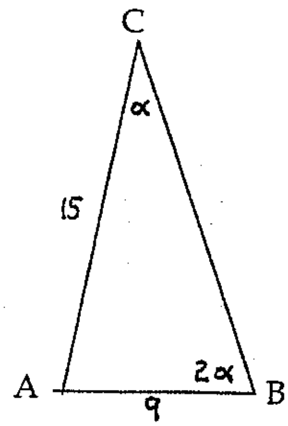
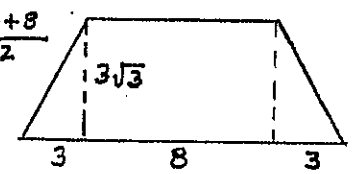
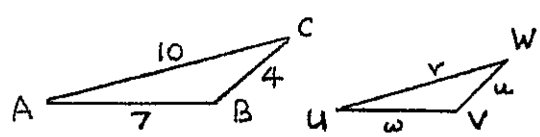


Figure 4

1. Area $= 3\sqrt{3}\,\frac{14+8}{2}$
$= 33\sqrt{3}$



2.



$u = k(4)$
$v = k(10)$
$+\ w = k(7)$
$15 = 21k$

$k = \frac{15}{21} = \frac{5}{7}$

$v = \frac{5}{7}(10) = \frac{50}{7}$

3. Let $\alpha = \angle APU$
As shown in the figure
$\angle UPB = \angle QPA = 60 - \alpha$
Let PQ intersect AF at V
$\Delta VPA \cong \Delta UPB$
Area(VPUA) $= a(\Delta VPA) + a(\Delta APU)$
$= a(\Delta UPB) + a(\Delta APU)$
$= a(\Delta APB) = \frac{s^2\sqrt{3}}{4}$

4. Let $\alpha = m\angle C$
$\frac{\sin\alpha}{9} = \frac{\sin 2\alpha}{15} = \frac{2\sin\alpha\cos\alpha}{15}$ so $\cos\alpha = \frac{5}{6}$
Let $x = BC$ and use the law of cosines
$81 = 225 + x^2 - 2(15)x\left(\frac{5}{6}\right) = 225 + x^2 - 25x$
$x^2 - 25x + 144 = (x-9)(x-16) = 0$. Show $x \neq 9$.
$\therefore\ x = 16$. For Heron's formula, $s = \frac{9+15+16}{2} = 20$
Area $= \sqrt{(20)(11)(4)(5)} = 20\sqrt{11}$

Solutions

Minnesota State High School Mathematics League

Individual Event

2005-06 Event 5C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

Give credit for $112a^5$

112 1. In the expansion of $\left(\frac{1}{2}a+4\right)^8$, what will be the coefficient of a^5?

$\frac{47}{85}$ or .0552 2. If you draw three cards from an ordinary 52 card deck, what is the probability of drawing at least one of the twelve face cards (i.e. a Jack, Queen, or King)?

72 3. Five pennies made in 2000, 2002, 2004, 2005, and 2006 are to be put in a pile, face up. In how many ways can this be done so that the 2005 and 2006 pennies are not touching each other?

.026 or $\frac{21875}{839,808}$ 4. Sarah was sent to get 8 cans of soda to have on hand for the study session. When she got to the machine, she found that she had six choices. Remembering that she had a die in her purse for some homework for her probability class, she decided to roll it 8 times, choosing the first flavor if a 1 came up, etc. for all six possibilities. Using this scheme, what is the probability that she gets exactly four diet cokes?

1. $\binom{8}{3}\left(\frac{a}{2}\right)^5(4)^3 = \frac{8!}{5!\,3!}\,\frac{1}{2^5}\left(2^2\right)^3a^5$
$= 112a^5$

2. There are $\binom{52}{3} = \frac{(52)(51)(50)}{3!}$ ways to draw three cards. Of these, $\binom{40}{3} = \frac{(40)(39)(38)}{3!}$ will contain no face cards.

$$\text{prob (no face cards)} = \frac{\frac{(40)(39)(38)}{3!}}{\frac{(52)(51)(50)}{3!}} = \frac{38}{85}$$

$$\text{prob (at least one face card)} = 1 - \frac{38}{85} = \frac{47}{85}$$

3. Temporarily glue the 2005 penny on top the 2006 penny. You can now make 4! piles; and with the 2006 penny glued on top the 2005 penny, you can make another 4! piles. These are piles not allowed, however, so there are $5! - 2(4!) = 72$ piles that are allowed.

4. Consider this a binomial probability in which the probability of success (a diet coke) is $\frac{1}{6}$; that of failure is $\frac{5}{6}$. The probability of exactly four successes is $\binom{8}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^4 = \frac{21875}{839808} = .026$

Solutions

Minnesota State High School Mathematics League

Individual Event

2005-06 Event 5D

Questions in this event are written, with permission, as variations of problems from the 2005 AMC-12 Exam. Review of this exam is excellent preparation, not only for this event, but for the 2006 AMC-12, which we strongly encourage you to take.

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

A = 1, B = 8 ____ 1. A and B are digits such that $(100A+10A+B)(2B+1)=2006$. Find A and B.

17 ____ 2. The mean average of the eighteen numbers in the set $A=\{x_1, \ldots x_{18}\}$ is 12. The mean average of the thirty numbers in the set $B=\{y_1, \ldots y_{30}\}$ is 20. What is the mean average of the forty-eight numbers in the set $A \cup B$?

9 ____ 3. A lattice point is a point having integers for coordinates. The point (4,5) is, for instance, a lattice point that lies on the line $7x-11y+27=0$. What is the total number of lattice points on this line that lie in the square

$$S=\{(x,y): 0 \le x \le 100, 0 \le y \le 100\}$$

4 ____ 4. Given:

$$f(x)=\begin{cases} 2x & 0 \le x \le 1 \\ 4-2x & 1 \le x \le 2 \end{cases}$$

$$g(x)=f(f(x))$$

$$h(x)=g(g(x))$$

Find $h\left(\frac{7}{8}\right)+h\left(\frac{1}{8}\right)$

4. $h\left(\frac{7}{8}\right) = g\left(g\left(\frac{7}{8}\right)\right)$

$g\left(\frac{7}{8}\right) = f\left(f\left(\frac{7}{8}\right)\right) = f\left(\frac{7}{4}\right) = \frac{1}{2}$

$h\left(\frac{7}{8}\right) = g\left(\frac{1}{2}\right) = f\left(f\left(\frac{1}{2}\right)\right) = f(1) = 2$

$h\left(\frac{1}{8}\right) = g\left(g\left(\frac{1}{8}\right)\right)$

$g\left(\frac{1}{8}\right) = f\left(f\left(\frac{1}{8}\right)\right) = f\left(\frac{1}{4}\right) = \frac{1}{2}$

$h\left(\frac{1}{8}\right) = g\left(\frac{1}{2}\right) = 2$

$h\left(\frac{7}{8}\right) + h\left(\frac{1}{8}\right) = 2 + 2 = 4$

1. $2006 = 2 \cdot 17 \cdot 59$

$2 \cdot 17$ is only two digits

$17 \cdot 59$ is four digits.

$\therefore\ 2 \cdot 59 = 118 = 100(1) + 10(1) + 8$

$A = 1$, $B = 8$. Ck. $2 \cdot 8 + 1 = 17$

3. $y = \frac{7}{11}x + \frac{27}{11}$ has a slope of $\frac{7}{11}$

The equation can be written $y-5 = \frac{7}{11}(x-4)$

from which we see that a parametric form is $x = 4 + 11t$, $y = 5 + 7t$.

$t = 0, 1, \ldots, 8$ all give lattice points in S.

2. $$\frac{(x_1+\cdots+x_{18})+(y_1+\cdots+y_{30})}{48} = \frac{18(12)+30(20)}{48} = 17$$

Answers

Minnesota State High School Mathematics League

Team Event

2005-06 Meet 5

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

a	b	c
50	25	375
2	1	3
8	4	24
18	9	81
32	16	192

1. A set of integer solutions to $a^3 + b^3 = c^2$ is given by $a = 50$, $b = 25$, and $c = 375$. There are other sets of solutions, some using smaller integers. You get 1 point, up to a maximum of 4, for each distinct set of integer solutions, none of the integers 0, that you obtain different from the one given.

84 2. How many integers are there between 100,000 and 1,000,000 that, like 134,789 consist of distinct digits that increase from left to right?

$\frac{3\sqrt{3} - 4}{2}$ OR .598 3. An equilateral ΔABC has sides of length 3. Point P is located interior to the triangle at a distance 1 from AB and a distance 1 from AC. Correct to three places to the right of the decimal, what is the distance of P from BC?

99 4. Expressing numbers to base ten (i.e. no trickery here), how many 1's will appear in the integer which is the sum of the ninety-nine integers $9 + 99 + 999 + \ldots + \underbrace{99......9}_{\text{ninety-nine 9's}}$

\$39 5. Albert, Bob and Charles play a game in which there is one clear loser. Starting with different amounts of money, they decide that the loser is to pay to each of the others an amount that doubles each winners money. After three games each man has lost exactly once, and each has \$24. Albert had the most money when they started. How much?

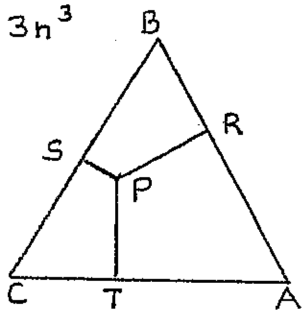
$\frac{r^2(m^2+1)}{-2m}$ 6. A line tangent to $x^2 + y^2 = r^2$ in the first quadrant has a slope of m. What, in terms of r and m, is the area of the triangle that this line forms with the positive coordinate axes?

Give three points if the only error is to omit the – sign.

Solutions to Team Event 5

1. Since powers of 2 and 3 appear, think of ways to involve 6. Eventually, try setting $a = 2n^2$, $b = n^2$
$a^3 + b^3 = (2n^2)^3 + (n^2)^3 = 9n^6 = (3n^3)^2$

n	a	b	$c = 3n^3$
1	2	1	3
2	8	4	24
3	18	9	81
4	32	16	192
5	50	25	375



2. Since 0 cannot lead, and it cannot follow given that the digits must increase from left to right, 0 cannot be used. From $\{1,2,3,4,5,6,7,8,9\}$ choose any six digits. Arrange them in order to get a solution. This can be done in $\binom{9}{6} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$ ways

3. Refer to the figure above. The distances from P to AB and to AC are $PR = PT = 1$. The distance from P to BC is $x = PS$.
Area $(\triangle ABC) = \frac{3^2\sqrt{3}}{4}$
$= \text{Area}(\triangle ABP) + \text{Area}(\triangle CAP) + \text{Area}(\triangle BCP)$

$\frac{9\sqrt{3}}{4} = \frac{1}{2}(3)(1) + \frac{1}{2}(3)(1) + \frac{1}{2}(3)x$

$\frac{3}{2}x = \frac{9\sqrt{3}}{4} - 3$; $x = 2 \cdot \frac{3\sqrt{3} - 4}{4} = .598$

4. $(10-1) + (10^2-1) + (10^3-1) + \cdots + (10^{99}-1)$

$= 10 + 10^2 + 10^3 + \cdots + 10^{99} - 99$

$= \underbrace{11\ldots11}_{\text{ninety-nine 1's}}0 - 99 = \underbrace{11\ldots11}_{\text{ninety-seven 1's}}000 + 110 - 99$

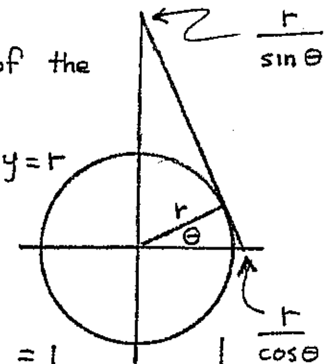
$= \underbrace{11\ldots1011}_{\text{ninety-nine 1's}}$

5. Work backwards, supposing that C loses the 3rd game. After paying A and B, each has 24; so each must have had 12. Make a table.

Games Complete	A	B	C	
3	24	24	24	C lost
2	12	12	48	B lost
1	6	42	24	A lost
0	39	21	12	

6. Normal form of the line is

$(\cos\theta)x + (\sin\theta)y = r$



The intercept form then is

$$\frac{x}{\left(\frac{r}{\cos\theta}\right)} + \frac{y}{\left(\frac{r}{\sin\theta}\right)} = 1$$

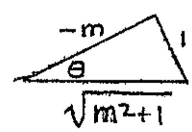
Area $= \frac{1}{2} \frac{r}{\cos\theta} \cdot \frac{r}{\sin\theta}$

Slope of the line $= m = \tan\alpha$

$\tan(90 - \theta) = -\tan\alpha = -m$

$\cot\theta = -m$

$\sin\theta = \frac{1}{\sqrt{m^2+1}}$



$\cos\theta = \frac{-m}{\sqrt{m^2+1}}$

Area $= \frac{r^2}{2} \frac{\sqrt{m^2+1}}{-m}$